

Durham Research Online

Deposited in DRO:

06 June 2017

Version of attached file:

Accepted Version

Peer-review status of attached file:

Peer-reviewed

Citation for published item:

Picouleau, C. and Paulusma, D. and Ries, B. (2017) 'Reducing the chromatic number by vertex or edge deletions.', *Electronic notes in discrete mathematics.*, 62 . pp. 243-248.

Further information on publisher's website:

<https://lipn.univ-paris13.fr/Lagos2017/>

Publisher's copyright statement:

© 2017 This manuscript version is made available under the CC-BY-NC-ND 4.0 license
<http://creativecommons.org/licenses/by-nc-nd/4.0/>

Additional information:

Use policy

The full-text may be used and/or reproduced, and given to third parties in any format or medium, without prior permission or charge, for personal research or study, educational, or not-for-profit purposes provided that:

- a full bibliographic reference is made to the original source
- a [link](#) is made to the metadata record in DRO
- the full-text is not changed in any way

The full-text must not be sold in any format or medium without the formal permission of the copyright holders.

Please consult the [full DRO policy](#) for further details.

Reducing the Chromatic Number by Vertex or Edge Deletions

Christophe Picouleau ^{a,1} Daniël Paulusma ^{b,2} Bernard Ries ^{c,3}

^a *Laboratoire CEDRIC
CNAM
Paris, France*

^b *School of Engineering and Computing Sciences
Durham University
Durham, United Kingdom*

^c *Department of Informatics
University of Fribourg
Fribourg, Switzerland*

Abstract

A vertex or edge in a graph is critical if its deletion reduces the chromatic number of the graph by 1. We consider the problems of testing whether a graph has a critical vertex or edge, respectively. We give a complete classification of the complexity of both problems for H -free graphs, that is, graphs with no induced subgraph isomorphic to H . Moreover, we show that an edge is critical if and only if its contraction reduces the chromatic number by 1. Hence, we obtain the same classification for the problem of testing if a graph has an edge whose contraction reduces the chromatic number by 1. As a consequence of our results, we are also able to complete the complexity classification of the more general vertex deletion and edge contraction blocker problems for H -free graphs when the graph parameter is the chromatic number.

Keywords: edge contraction, vertex deletion, chromatic number.

1 Introduction

A vertex or edge of a graph G is *critical* if its removal reduces the chromatic number $\chi(G)$ by 1. An edge is *contraction-critical* if its contraction reduces $\chi(G)$ by 1. We call the problems of deciding if a graph has a critical vertex, critical edge or contraction-critical edge CRITICAL VERTEX, CRITICAL EDGE and CONTRACTION-CRITICAL EDGE, respectively. It is not difficult to show that these three problems are computationally hard in general. Here we classify the computational complexity of them for graphs with no induced subgraph isomorphic to some specified graph H ; we call such graphs H -free.

Each of the above decision problems can be generalized as follows. Let S be a fixed set of one or more graph operations, and let π be some fixed graph parameter. Given a graph G , and an integer k , we ask if G can be modified into a graph G' by using at most k operations from S so that $\pi(G') \leq \pi(G) - d$ for some given *threshold* $d \geq 0$. Such problems are called *blocker problems*, as the vertices or edges involved “block” some desirable graph property (such as being colorable with only a few colors). Over the last few years, blocker problems have been well studied, see for instance [1,2,3,4,5,9,10,11,12]. In these papers the set S consists of a single operation that is either a vertex deletion vd , edge deletion ed , or edge contraction ec . The decision problems are called VERTEX DELETION BLOCKER(π) if $S = \{\text{vd}\}$, EDGE DELETION BLOCKER(π) if $S = \{\text{ed}\}$ and CONTRACTION BLOCKER(π) if $S = \{\text{ec}\}$. By taking $d = k = 1$ and $\pi = \chi$ we obtain the problems CRITICAL VERTEX, CRITICAL EDGE and CONTRACTION-CRITICAL EDGE. The complexities of VERTEX DELETION BLOCKER(χ) and CONTRACTION BLOCKER(χ) are known for H -free graphs if H is connected [10]. As a consequence of our results for $k = d = 1$, we can complete these two classifications for all graphs H , just as we did in a previous paper [11] for $\pi = \alpha$ (independence number) and $\pi = \omega$ (clique number), except for the case when $\pi = \omega$, $S = \{\text{ec}\}$ and $H = C_3 + P_1$. The EDGE DELETION BLOCKER(χ) problem is known [1] to be polynomial-time solvable for threshold graphs and NP-hard for cobipartite graphs. For this problem we obtain a partial classification that leaves exactly two cases open.

Terminology. The graph $G + G'$ is the disjoint union of the graphs G and G' . The graph pG is the disjoint union of p copies of G . We let K_n , P_n and C_n be the complete graph, path and cycle on n vertices, respectively. For a subset

¹ christophe.picouleau@cnam.fr

² daniel.paulusma@durham.ac.uk

³ bernard.ries@unifr.ch

$S \subseteq V$ of a graph G , we let $G[S]$ be the subgraph of G induced by S . We write $H \subseteq_i G$ if H is an induced subgraph of G . The graph \overline{G} is the complement of G . A graph G is (H_1, \dots, H_p) -free if G is H -free for every $H \in \{H_1, \dots, H_p\}$. The contraction of an edge uv removes u and v from V and replaces them by a new vertex made adjacent to precisely those vertices adjacent to u or v in G .

2 Critical Vertices and Edges

We start with the following result (proof omitted).

Proposition 2.1 *An edge is critical if and only if it is contraction-critical.*

Due to Proposition 2.1, CRITICAL EDGE and CONTRACTION-CRITICAL EDGE have the same classification for H -free graphs.

Let COLORING be the problem of deciding whether $\chi(G) \leq k$ for some given integer $k \geq 1$. We need the following known result.

Theorem 2.2 ([7]) *If a graph $H \subseteq_i P_4$ or of $H \subseteq_i P_1 + P_3$, then COLORING is polynomial-time solvable for H -free graphs, otherwise it is NP-complete.*

We also need the following lemma.

Lemma 2.3 ([10]) *Let H be a graph. Then CRITICAL VERTEX and CRITICAL EDGE are NP-hard for H -free graphs if $H \supseteq_i K_{1,3}$ or $H \supseteq_i C_r$ for some $r \geq 3$.*

The clique covering number $\sigma(G)$ of a graph G is the smallest number of cliques in a graph so that each vertex belongs to exactly one clique. The hardness construction in the (omitted) proof of our next result uses clique covers and some other elements of the proof of Theorem 2 in [7] for showing that COLORING is NP-hard for $(C_5, 4P_1, P_1 + 2P_2, 2P_2)$ -free graphs.

Theorem 2.4 *CRITICAL VERTEX and CRITICAL EDGE are both co-NP-hard for $(C_5, 4P_1, 2P_1 + P_2, 2P_2)$ -free graphs.*

We can now prove the following dichotomies.

Theorem 2.5 *Let H be a graph. Then CRITICAL VERTEX, CRITICAL EDGE and CONTRACTION-CRITICAL EDGE restricted to H -free graphs are polynomial-time solvable if $H \subseteq_i P_1 + P_3$ or $H \subseteq_i P_4$, and NP-hard or co-NP-hard otherwise.*

Proof. Let $H \subseteq_i P_1 + P_3$ or $H \subseteq_i P_4$. Let G be an H -free graph. By Theorem 2.2 we can compute $\chi(G)$ in polynomial time. We note that any vertex deletion results in a graph that is H -free as well. Hence in order to solve CRITICAL VERTEX we can compute the chromatic number of $G - v$ for

each vertex v in polynomial time and compare it with $\chi(G)$. As $(P_1 + P_3)$ -free graphs and P_4 -free graphs are closed under edge contraction as well, we can follow the same approach for solving CONTRACTION-CRITICAL EDGE. By Proposition 2.1 we obtain the same result for CRITICAL EDGE.

Now suppose that neither $H \subseteq_i P_1 + P_3$ nor $H \subseteq_i P_4$. By Proposition 2.1 it suffices to consider CRITICAL VERTEX and CRITICAL EDGE. If H has a cycle or an induced claw, then we use Lemma 2.3. Assume not. Then H is a disjoint union of r paths for some $r \geq 1$. If $r \geq 4$ we use Theorem 2.4. If $r = 3$ then either $H = 3P_1 \subseteq_i P_1 + P_3$, which is not possible, or $H \supseteq_i 2P_1 + P_2$ meaning that we can apply Theorem 2.4 again. Suppose $r = 2$. If both paths contain an edge, then $2P_2 \subseteq_i H$. If at most one path has edges, then it must have at least four vertices, as otherwise $H \subseteq_i P_1 + P_3$. This means that $2P_1 + P_2 \subseteq_i H$. In both cases we apply Theorem 2.4. If $r = 1$, then H is a path on at least five vertices, which means $2P_2 \subseteq_i H$. We apply Theorem 2.4 again. \square

3 Vertex Deletion and Contraction Blocker Problems

We need the following two results from our previous papers as lemmas.

Lemma 3.1 ([5]) *CONTRACTION BLOCKER(χ) and VERTEX DELETION BLOCKER(χ) are polynomial-time solvable for P_4 -free graphs.*

Lemma 3.2 ([10]) *For $3P_1$ -free graphs, CONTRACTION BLOCKER(χ) is NP-hard, but VERTEX DELETION BLOCKER(χ) problem is polynomial-time solvable.*

A graph G is *complete multipartite* if $V(G)$ can be partitioned into k independent sets V_1, \dots, V_k for some integer k , such that two vertices are adjacent if and only if they belong to two different sets V_i and V_j . The graph $\overline{P_1 + P_3}$ is also known as the *paw*. We need a result of Olariu on paw-free graphs.

Lemma 3.3 ([8]) *Every connected $\overline{P_1 + P_3}$ -free graph is either triangle-free or complete multipartite.*

Two disjoint subsets of vertices in a graph are *complete* if there is an edge between every vertex of A and every vertex of B . Lemma 3.3 implies the following lemma, which we use to prove Proposition 3.5 (proofs omitted).

Lemma 3.4 *The vertex set of every $(P_1 + P_3)$ -free graph G can be decomposed into two disjoint sets A and B such that $G[A]$ is $3P_1$ -free, $G[B]$ is P_4 -free and A and B are complete to each other.*

Proposition 3.5 *VERTEX DELETION BLOCKER(χ) problem is polynomial-time solvable for $(P_1 + P_3)$ -free graphs.*

We can now state the following two dichotomies. The first dichotomy was shown in [10] already. The tractable cases of the second dichotomy follow from Lemmas 3.1 and Proposition 3.5, and the hard cases from Theorem 2.5. Note that due to Lemma 3.2 the two dichotomies are not the same.

Theorem 3.6 *Let H be a graph. Then the following holds:*

- *If $H \subseteq_i P_4$, then $\text{CONTRACTION BLOCKER}(\chi)$ for H -free graphs is polynomial-time solvable for H -free graphs, and it is **NP**-hard otherwise.*
- *If $H \subseteq_i P_1 + P_3$ or P_4 , then $\text{VERTEX DELETION BLOCKER}(\chi)$ for H -free graphs is polynomial-time solvable, and it is **NP**-hard or co-**NP**-hard otherwise.*

4 Conclusions

We gave complete complexity classifications of five problems CRITICAL VERTEX , CRITICAL EDGE , $\text{CONTRACTION-CRITICAL EDGE}$, $\text{VERTEX DELETION BLOCKER}(\chi)$ and $\text{CONTRACTION BLOCKER}(\chi)$ for H -free graphs. The classifications for the first four problems coincide with the known classification of COLORING for H -free graphs (see Theorem 2.2), whereas the case $H = P_1 + P_3$ is no longer tractable for the latter problem. We finish our paper with two directions for future work.

First, we have no full complexity classification for the $\text{EDGE DELETION BLOCKER}(\chi)$ problem for H -free graphs. This problem has been less studied than the vertex deletion and edge contraction variant. The reason for this is that H -free graphs are not closed under taking edge deletions, whereas they are closed under vertex deletions and in case when H is a linear forest under edge contractions as well. Proposition 2.1 combined with Lemma 2.3 shows that the problem is **NP**-hard if H is not the disjoint union of paths. Bazgan et al. [1] showed that $\text{EDGE DELETION BLOCKER}(\chi)$ is polynomial-time solvable for threshold graphs, that is, for $(C_4, 2P_2, P_4)$ -free graphs, and **NP**-hard for cobipartite graphs, that is, for complements of bipartite graphs. The latter class is a subclass of the class of $3P_1$ -free graphs. The above combined with Theorem 2.4 leads to a classification of $\text{EDGE DELETION BLOCKER}(\chi)$ for H -free graphs up to two open cases, namely when $H = P_1 + P_2$ or $H = P_4$.

The second research direction is to classify the complexity of these six problems for graphs classes characterized by a family $\{H_1, \dots, H_p\}$ of graphs for any $p \geq 2$. We note that such a classification for COLORING is still wide open even for $p = 2$ (see [6]). Hence, research in this direction might lead to an increased understanding of the complexity of the COLORING problem.

References

- [1] Bazgan, C., C. Bentz, C. Picouleau and B. Ries, *Blockers for the stability number and the chromatic number*, Graphs and Combinatorics **31** (2015), 73–90.
- [2] Bazgan, C., S. Toubaline and Z. Tuza, *The most vital nodes with respect to independent set and vertex cover*, Discrete Applied Mathematics **159** (2011), 1933–1946.
- [3] Bentz, C., M.-C. Costa, D. de Werra, C. Picouleau and B. Ries, *Weighted transversals and blockers for some optimization problems in graphs*, Progress in Combinatorial Optimization, Wiley-ISTE, 2012.
- [4] Costa, M.-C., D. de Werra, C. Picouleau, *Minimum d -blockers and d -transversals in graphs*, Journal of Combinatorial Optimization **22** (2011,) 857–872.
- [5] Diner, O., D. Paulusma, C. Picouleau and B. Ries, *Contraction blockers for graphs with forbidden induced paths*, Proc. CIAC 2015, Lecture Notes in Computer Science **9079** (2015), 194–207.
- [6] Golovach, P.A., M. Johnson, D. Paulusma and J. Song, *A survey on the computational complexity of colouring graphs with forbidden subgraphs*, Journal of Graph Theory **84** (2017), 331–363.
- [7] Král', D., J. Kratochvíl, Z. Tuza, and G.J. Woeginger, *Complexity of coloring graphs without forbidden induced subgraphs*, Proc. WG 2001, Lecture Notes in Computer Science **2204** (2001), 254–262.
- [8] Olariu, S., *Paw-free graphs*, Information Processing Letters **28** (1988), 53–54.
- [9] Pajouh, F.M., V. Boginski and E. L. Pasiliao, *Minimum vertex blocker clique problem*, Networks **64** (2014), 48–64.
- [10] Paulusma, D., C. Picouleau and B. Ries, *Reducing the clique and chromatic number via edge contractions and vertex deletions*, Proc. ISCO 2016, Lecture Notes in Computer Science **9849** (2016), 38–49.
- [11] Paulusma, D., C. Picouleau and B. Ries, *Blocking independent sets for H -free graphs via edge contractions and vertex deletions*, Proc. TAMC 2017, Lecture Notes in Computer Science **10185** (2017), 470–483.
- [12] Ries, B., C. Bentz, C. Picouleau, D. de Werra, M.-C. Costa and R. Zenklusen, *Blockers and transversals in some subclasses of bipartite graphs: when caterpillars are dancing on a grid*, Discrete Mathematics **310** (2010), 132–146.